

APPROXIMATION PROPERTIES DETERMINED BY OPERATOR IDEALS

Sonia Berrios and Geraldo Botelho*

Abstract

Given an operator ideal \mathcal{I} , a Banach space E has the \mathcal{I} -approximation property if operators on E can be uniformly approximated on compact subsets of E by operators belonging to \mathcal{I} . In this paper the \mathcal{I} -approximation property is studied in projective tensor products, spaces of linear functionals, spaces of homogeneous polynomials (in particular, spaces of linear operators), spaces of holomorphic functions and their preduals.

1 Introduction

Given Banach spaces E and F , by $\mathcal{L}(E; F)$ we denote the Banach space of all bounded linear operators from E to F endowed with the usual operator sup norm. The subspaces of $\mathcal{L}(E; F)$ formed by all finite rank, compact and weakly compact operators are denoted by $\mathcal{F}(E; F)$, $\mathcal{K}(E; F)$ and $\mathcal{W}(E; F)$, respectively. For a subset S of $\mathcal{L}(E; F)$, the symbol \overline{S}^{τ_c} represents the closure of S with respect to the compact-open topology τ_c .

It is well known that a Banach space E has

- the approximation property (in short, E has AP) if $\mathcal{L}(E; E) = \overline{\mathcal{F}(E; E)}^{\tau_c}$,
- the compact approximation property (in short, E has CAP) if $\mathcal{L}(E; E) = \overline{\mathcal{K}(E; E)}^{\tau_c}$,
- the weakly compact approximation property (in short, E has WCAP) if $\mathcal{L}(E; E) = \overline{\mathcal{W}(E; E)}^{\tau_c}$.

The AP is a classic in Banach space theory (see [13]) and is one of the main subjects of Grothendieck [32]. The CAP has been more studied in the last decades and recently (see, e.g. [14, 15, 18, 19]), but it goes back to Banach [4, p. 237]. The WCAP has been studied more recently (see [19, 20]). Having in mind that \mathcal{F}, \mathcal{K} and \mathcal{W} are operator ideals, the properties above can be regarded as particular instances of the following general concept:

*Supported by CNPq Grant 306981/2008-4.
2010 Mathematics Subject Classification: 46B28, 47B10, 46G20, 46G25.

Definition 1.1. Let \mathcal{I} be an operator ideal. A Banach space E is said to have the \mathcal{I} -approximation property (in short, E has \mathcal{I} -AP) if $\mathcal{L}(E; E) = \overline{\mathcal{I}(E; E)}^{\tau_c}$.

Several variants of the approximation property have been studied recently (see, e.g., [14, 16, 22, 24, 39, 48, 53]), including ones closely related to the \mathcal{I} -AP [38, 41, 47].

The selection of operator ideals instead of other classes of linear operators related to \mathcal{F} , \mathcal{K} and \mathcal{W} is justified by the fact that even the most basic results depend on the ideal property (cf. Section 3).

It is clear that if E has AP then E has \mathcal{I} -AP for every operator ideal \mathcal{I} . In particular, Banach spaces with Schauder basis (e.g., ℓ_p , $1 \leq p < \infty$, and c_0) have \mathcal{I} -AP for every operator ideal \mathcal{I} .

Let us stress that different ideals may give rise to different approximation properties: (i) Willis [55] showed that there are spaces with CAP but not with AP; (ii) Szankowski [54] proved that for $1 \leq p < 2$, ℓ_p has a subspace S_p without CAP, so $S_{\frac{3}{2}}$ has WCAP but not CAP and S_1 has $\mathcal{CC} \cap \mathcal{C}_2$ -AP but not CAP, where \mathcal{CC} and \mathcal{C}_2 are the ideals of completely continuous and cotype 2 operators, respectively. The fact that different operator ideals usually give rise to different approximation properties justifies the study of the \mathcal{I} -AP for arbitrary operator ideals, which is the aim of this paper. In Example 3.3 we shall see that different ideals may generate the same approximation property.

The study of the approximation property and its already studied variants is very rich and multifaceted, so the study of the \mathcal{I} -AP could follow several different trends. This means that, to study the \mathcal{I} -AP, choices have to be made. In this paper we have chosen to study the \mathcal{I} -AP in projective tensor products (Section 5) and in spaces of mappings between Banach spaces, namely, spaces of linear functionals (Section 4), spaces of homogeneous polynomials (Section 6) and spaces of holomorphic functions and their preduals (Section 7). Proposition 6.6 fixes and generalizes a result of [20].

The results we prove in the different sections of the paper seem - at first glance - to be completely disconnected. Connections of results from different sections are given in Section 7.

2 Notation and preliminaries

When F is the scalar field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we shall write E' instead of $\mathcal{L}(E; \mathbb{K})$. The *compact-open topology* or *topology of compact convergence* is the locally convex topology τ_c on $\mathcal{L}(E; F)$ which is generated by the seminorms of the form

$$p_K(T) = \sup_{x \in K} \|T(x)\|,$$

where K ranges over all compact subsets of E .

Given a subset S of $\mathcal{L}(E; F)$, $\overline{S}^{\tau_c} = \mathcal{L}(E; F)$ if and only if for every $T \in$

$\mathcal{L}(E; F)$, every compact set $K \subseteq E$ and every $\varepsilon > 0$, there is an operator $U \in S$ such that $\|T(x) - U(x)\| < \varepsilon$ for every $x \in K$.

An *operator ideal* \mathcal{I} is a subclass of the class of all continuous linear operators between Banach spaces such that for all Banach spaces E e F , the component $\mathcal{I}(E; F) = \mathcal{L}(E; F) \cap \mathcal{I}$ satisfy:

- (a) $\mathcal{I}(E; F)$ is a linear subspace of $\mathcal{L}(E; F)$ which contains the finite rank operators.
- (b) Ideal property: If $T \in \mathcal{L}(E; F)$, $R \in \mathcal{I}(F; G)$ and $S \in \mathcal{L}(G; H)$, then the composition $S \circ R \circ T$ is in $\mathcal{I}(E; H)$.

By id_E we mean the identity operator on the Banach space E . For a given operator ideal \mathcal{I} , by $\overline{\mathcal{I}}$ we mean the closure of \mathcal{I} , that is, $\overline{\mathcal{I}}(E; F) = \overline{\mathcal{I}(E; F)}$ for every Banach spaces E and F . For the theory of operator ideals we refer to [50, 21]. Here is a list of the operator ideals occurring in this paper:

\mathcal{F} = finite rank operators (the range is finite-dimensional),

$\mathcal{A} := \overline{\mathcal{F}}$ = approximable operators,

\mathcal{K} = compact operators (bounded sets are mapped onto relatively compact sets),

\mathcal{W} = weakly compact operators (bounded sets are mapped onto relatively weakly compact sets),

\mathcal{CC} = completely continuous operators (weakly convergent sequences are sent to norm convergent sequences),

\mathcal{N}_p = p -nuclear operators,

\mathcal{C}_p = cotype p operators,

\mathcal{T}_p = type p operators,

\mathcal{D} = dualisable operators,

\mathcal{S} = separable operators (the range is separable),

$\mathcal{DP} := \mathcal{W}^{-1} \circ \mathcal{CC}$ = Dunford-Pettis operators,

\mathcal{J} = integral operators,

\mathcal{SN} = strongly nuclear operators,

\mathcal{SS} = strictly singular operators (restrictions to infinite-dimensional subspaces are never isomorphisms),

\mathcal{SC} = strictly cosingular operators,

Π_p = absolutely p -summing operators,

$\Pi_{r,p,q}$ = absolutely (r, p, q) -summing operators [50, 17.1],

Γ_p = p -factorable operators,

\mathcal{KC} = K-convex operators [21, 31.1],

\mathcal{QN} = quasinuclear operators [21, Ex. 9.13],

$\mathcal{L}_{\infty,q,\gamma}$ = Lorentz-Zigmund operators [17],

\mathcal{U}_p = operators having approximation numbers belonging to ℓ_p [50, 14.2.4],

$\mathcal{L}_{p,q}$ = (p, q) -factorable operators,

\mathcal{K}_p = p -compact operators (bounded sets are mapped to relatively p -compact sets),

\mathcal{QN}_p = quasi p -nuclear operators [23].

3 Basic results

The results of this section, except for some implications of Proposition 3.4, are elementary enough to have their proofs omitted. Nevertheless, it is worth mentioning that the ideal property plays a crucial role in their (easy) proofs.

The following characterizations are simple but useful.

Proposition 3.1. *Given an operator ideal \mathcal{I} , the following are equivalent for a Banach space E :*

- (a) *E has the \mathcal{I} -approximation property.*
- (b) *E has the $\overline{\mathcal{I}}$ -approximation property.*
- (c) *$\text{id}_E \in \overline{\mathcal{I}(E; E)}^{\tau_c}$.*
- (d) *For each compact set $K \subseteq E$ and every $\varepsilon > 0$, there is an operator $T \in \mathcal{I}(E; E)$ such that $\|T(x) - x\| < \varepsilon$ for every $x \in K$.*

Given operator ideals \mathcal{I}_1 and \mathcal{I}_2 , we say that $\mathcal{I}_1\text{-AP} = \mathcal{I}_2\text{-AP}$ if the Banach spaces having $\mathcal{I}_1\text{-AP}$ are exactly the ones having $\mathcal{I}_2\text{-AP}$. The equivalence between (a) and (b) in Proposition 3.1 says that $\mathcal{I}\text{-AP} = \overline{\mathcal{I}}\text{-AP}$ for every operator ideal \mathcal{I} . In particular,

Corollary 3.2. *Let \mathcal{I}_1 and \mathcal{I}_2 be operator ideals. If $\overline{\mathcal{I}_1} = \overline{\mathcal{I}_2}$, then $\mathcal{I}_1\text{-AP} = \mathcal{I}_2\text{-AP}$.*

Example 3.3. Since $\mathcal{F} \subseteq \mathcal{N}_p \subseteq \overline{\mathcal{F}} = \mathcal{A}$ [35, Proposition 19.7.3], it holds that $\mathcal{N}_p\text{-AP} = \text{AP}$ whereas $\mathcal{F} \neq \mathcal{N}_p \neq \overline{\mathcal{F}} = \mathcal{A}$.

Let us see a few more interesting conditions that are equivalent to the $\mathcal{I}\text{-AP}$:

Proposition 3.4. *Let \mathcal{I} be an operator ideal. The following statements are equivalent for a Banach space E :*

- (a) *E has the \mathcal{I} -approximation property.*
- (b) *For every Banach F , $\mathcal{L}(E; F) = \overline{\mathcal{I}(E; F)}^{\tau_c}$.*
- (c) *For every Banach F , $\mathcal{L}(F; E) = \overline{\mathcal{I}(F; E)}^{\tau_c}$.*
- (d) *$\sum_{n=1}^{\infty} x'_n(T(x_n)) = 0$ for every $T \in \mathcal{L}(E; E)$ whenever the sequences $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ are such that $\sum_{n=1}^{\infty} \|x'_n\| \|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x'_n(T(x_n)) = 0$ for every $T \in \mathcal{I}(E; E)$.*
- (e) *$\sum_{n=1}^{\infty} x'_n(x_n) = 0$ whenever the sequences $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ are such that $\sum_{n=1}^{\infty} \|x'_n\| \|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x'_n(T(x_n)) = 0$ for every $T \in \mathcal{I}(E; E)$.*

Proof. (a) \implies (b) and (a) \implies (c) are straightforward. (b) \implies (a), (c) \implies (a) and (d) \implies (e) are obvious.

(e) \implies (a) Let $\varphi \in (\mathcal{L}(E; E), \tau_c)'$ be such that $\varphi(T) = 0$ for every $T \in \mathcal{I}(E; E)$. By Grothendieck's description [32] of the functionals belonging to $(\mathcal{L}(E; E), \tau_c)'$ (proofs can be found in [40, Proposition 1.e.3] and [27, Lemma VIII.3.3]), there are sequences $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ such that $\sum_{n=1}^{\infty} \|x'_n\| \|x_n\| < \infty$

∞ and $\varphi(T) = \sum_{n=1}^{\infty} x'_n(T(x_n))$ for every $T \in \mathcal{L}(E; E)$. By (d) we have that $\varphi(id_E) = \sum_{n=1}^{\infty} x'_n(x_n) = 0$. Hence $\varphi(id_E) = 0$ for every functional $\varphi \in (\mathcal{L}(E; E), \tau_c)'$ that vanishes on $\mathcal{I}(E; E)$. By the Hahn-Banach theorem (see, e.g., [43, Corollary 2.2.20]) it follows that $id_E \in \overline{\mathcal{I}(E; E)}^{\tau_c}$.

(a) \implies (d) Assume that (d) does not hold. In this case there are sequences $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ such that $\sum_{n=1}^{\infty} \|x'_n\| \|x_n\| < \infty$, $\sum_{n=1}^{\infty} x'_n(T(x_n)) = 0$ for every $T \in \mathcal{I}(E; E)$ and $\sum_{n=1}^{\infty} x'_n(U(x_n)) \neq 0$ for some $U \in \mathcal{L}(E; E)$. Defining

$$\varphi: \mathcal{L}(E; E) \longrightarrow \mathbb{K}, \quad \varphi(T) = \sum_{n=1}^{\infty} x'_n(T(x_n)),$$

by the above mentioned Grothendieck's description we know that $\varphi \in (\mathcal{L}(E; E), \tau_c)'$. Thus φ vanishes on $\mathcal{I}(E; E)$ but $\varphi(U) \neq 0$. Calling on Hahn-Banach once more we conclude that $U \notin \overline{\mathcal{I}(E; E)}^{\tau_c}$, which contradicts (a). \square

As expected, \mathcal{I} -AP is inherited by complemented subspaces and is stable under the formation of finite cartesian products:

Proposition 3.5. *Let \mathcal{I} be an operator ideal and E be a Banach space with the \mathcal{I} -approximation property. Then every complemented subspace of E has the \mathcal{I} -approximation property as well.*

Proposition 3.6. *Let \mathcal{I} be an operator ideal, $k \in \mathbb{N}$ and E_1, \dots, E_k be Banach spaces. Then the finite direct sum (or cartesian product) $E = \bigoplus_{n=1}^k E_n$ has the \mathcal{I} -approximation property if and only if E_1, \dots, E_n have the \mathcal{I} -approximation property.*

4 Duality

In this section we study the dual properties of the \mathcal{I} -approximation property. Given an operator ideal \mathcal{I} and Banach spaces E and F , define

$$\mathcal{I}^{dual}(E; F) = \{S \in \mathcal{L}(E; F) \text{ such that the adjoint operator } S' \in \mathcal{I}(F'; E')\}.$$

It is well known that \mathcal{I}^{dual} is an operator ideal. By J_E we mean the canonical embedding from E to E'' .

Proposition 4.1. *Let \mathcal{I}_1 and \mathcal{I}_2 be operator ideals. If E' has \mathcal{I}_2 -AP, F is reflexive and $\mathcal{I}_2(F'; E') \subseteq \mathcal{I}_1^{dual}(F'; E')$, then $\mathcal{L}(E; F) = \overline{\mathcal{I}_1(E; F)}^{\tau_c}$.*

Proof. Let $V \in \mathcal{L}(E; F)$ and let $\varphi \in (\mathcal{L}(E; F), \tau_c)'$ be such that $\varphi(T) = 0$ for every $T \in \mathcal{I}_1(E; F)$. It is enough to show that $\varphi(V) = 0$, because in this case $V \in \overline{\mathcal{I}_1(E; F)}^{\tau_c}$ by [43, Corollary 2.2.20]. Calling on Grothendieck's description of $(\mathcal{L}(E; F), \tau_c)'$ once more, there are sequences $(x_n) \subseteq E$ and $(y'_n) \subseteq F'$ such that $\sum_{n=1}^{\infty} \|y'_n\| \|x_n\| < \infty$ and $\varphi(U) = \sum_{n=1}^{\infty} y'_n(U(x_n))$. Let $S \in \mathcal{I}_2(F'; E')$.

By assumption we have that $S' \in \mathcal{I}_1(E''; F'')$. From the reflexivity of F we may define $R := (J_F)^{-1} \circ S' \circ J_E \in \mathcal{I}_1(E; F)$. For every $u \in F''$ and $v \in F'$,

$$\langle u, v \rangle = \langle J_F((J_F)^{-1}(u)), v \rangle = \langle v, (J_F)^{-1}(u) \rangle.$$

Observe that $\phi(\cdot) = \sum_n J_E(x_n)(\cdot)y'_n \in (\mathcal{L}(F'; E'), \tau_c)'$ and

$$\begin{aligned} \phi(S) &= \sum_{n=1}^{\infty} J_E(x_n)(S)y'_n = \sum_{n=1}^{\infty} J_E(x_n)(S(y'_n)) = \sum_{n=1}^{\infty} \langle J_E(x_n), S(y'_n) \rangle \\ &= \sum_{n=1}^{\infty} \langle S' \circ J_E(x_n), y'_n \rangle = \sum_{n=1}^{\infty} \langle y'_n, (J_F)^{-1} \circ S' \circ J_E(x_n) \rangle \\ &= \sum_{n=1}^{\infty} \langle y'_n, R(x_n) \rangle = \sum_{n=1}^{\infty} y'_n(R(x_n)) = 0. \end{aligned}$$

So $\phi(S) = 0$ for every $S \in \mathcal{I}_2(F'; E')$. Since E' has \mathcal{I}_2 -AP, $\mathcal{L}(F'; E') = \overline{\mathcal{I}_2(F'; E')}^{\tau_c}$ by Proposition 3.4. Therefore [43, Corollary 2.2.20] yields $\phi(V') = 0$. Thus

$$\begin{aligned} 0 &= \phi(V') = \sum_{n=1}^{\infty} J_E(x_n)(V'(y'_n)) = \sum_{n=1}^{\infty} \langle J_E(x_n), V'(y'_n) \rangle \\ &= \sum_{n=1}^{\infty} \langle V'(y'_n), x_n \rangle = \sum_{n=1}^{\infty} \langle y'_n, V(x_n) \rangle = \varphi(V). \end{aligned}$$

The proof is complete. \square

Theorem 4.2. *Let \mathcal{I}_1 and \mathcal{I}_2 be operator ideals such that either $\mathcal{I}_2 \subseteq \mathcal{I}_1^{dual}$ or $\mathcal{I}_2^{dual} \subseteq \mathcal{I}_1$ and E be a reflexive Banach space.*

- (a) *If E' has \mathcal{I}_2 -AP then E has \mathcal{I}_1 -AP.*
- (b) *If E has \mathcal{I}_2 -AP then E' has \mathcal{I}_1 -AP.*

Proof. Assume that $\mathcal{I}_2^{dual} \subseteq \mathcal{I}_1$. Let $u \in \mathcal{I}_2(E'; E')$. Since E is reflexive, $((J_E)^{-1} \circ u' \circ J_E)' = u \in \mathcal{I}_2(E'; E')$, hence $(J_E)^{-1} \circ u' \circ J_E \in \mathcal{I}_2^{dual}(E; E)$. By assumption we have $(J_E)^{-1} \circ u' \circ J_E \in \mathcal{I}_1(E; E)$. Then $u' = J \circ (J_E)^{-1} \circ u' \circ J_E \circ (J_E)^{-1} \in \mathcal{I}_1(E; E)$ by the ideal property, that is, $u \in \mathcal{I}_1^{dual}(E'; E')$. We have just proved that $\mathcal{I}_2(E'; E') \subseteq \mathcal{I}_1^{dual}(E'; E')$. Since this condition holds trivially if $\mathcal{I}_2 \subseteq \mathcal{I}_1^{dual}$, in both cases we have $\mathcal{I}_2(E'; E') \subseteq \mathcal{I}_1^{dual}(E'; E')$.

(a) Suppose that E' has \mathcal{I}_2 -AP. Since E is reflexive, we have $\mathcal{L}(E; E) = \overline{\mathcal{I}_1(E; E)}^{\tau_c}$ from Proposition 4.1, so E has \mathcal{I}_1 -AP.

(b) Suppose that E has \mathcal{I}_2 -AP. Since E and E'' are isometrically isomorphic, it follows that E'' has \mathcal{I}_2 -AP. Hence E' has \mathcal{I}_1 -AP by (a). \square

Corollary 4.3. *Let \mathcal{I} be an operator ideal such that either $\mathcal{I} \subseteq \mathcal{I}^{dual}$ or $\mathcal{I}^{dual} \subseteq \mathcal{I}$ and E be a reflexive Banach space. Then E' has the \mathcal{I} -approximation property if and only if E has the \mathcal{I} -approximation property.*

Given $1 \leq p < \infty$, p^* stands for the conjugate of p , that is $\frac{1}{p} + \frac{1}{p^*} = 1$. For the definition of the adjoint ideal \mathcal{I}^* of the operator ideal \mathcal{I} , see, e.g., [26, p. 132].

Example 4.4. Let us see that there is plenty of ideals satisfying the conditions of Theorem 4.2 and Corollary 4.3.

- (i) $\mathcal{N}_1^{dual} \subseteq \mathcal{J}$ [21, Ex. 16.9], $\mathcal{SS}^{dual} \subseteq \mathcal{SC}$ and $\mathcal{SC}^{dual} \subseteq \mathcal{SS}$ [25, 1.18], $\Gamma_p^{dual} = \Gamma_{p^*}$ [26, p. 186], $\Pi_1^{dual} = \Gamma_1^*$ [26, Corollary 9.5], $\mathcal{T}_p \subseteq \mathcal{C}_{p^*}^{dual}$ and $\mathcal{C}_{p^*} \circ \mathcal{KC} \subseteq \mathcal{T}_p^{dual}$ for $1 < p \leq 2$ [21, 31.2], $\mathcal{N}_1^{dual} \subseteq \mathcal{QN}$ [21, Ex. 9.13(b)], $\Pi_{r,p,q}^{dual} = \Pi_{r,q,p}$ [50, Theorem 17.1.5], $\mathcal{L}_{p,q}^{dual} = \mathcal{L}_{q,p}$ [12, p. 68], $\mathcal{K}_p = \mathcal{QN}_p^{dual}$ [23].
- (ii) The following ideals are completely symmetric (that is $\mathcal{I} = \mathcal{I}^{dual}$): $\mathcal{F}, \mathcal{A}, \mathcal{K}, \mathcal{W}$ [50, Proposition 4.4.7], \mathcal{J} [21, Corollary 10.2.2], \mathcal{SN} [35, Theorem 19.9.3], \mathcal{U}_p , $0 < p < \infty$ [50, Theorem 14.2.5] and \mathcal{KC} [21, 31.1].
- (iii) The following ideals satisfy $\mathcal{I} \subseteq \mathcal{I}^{dual}$: \mathcal{N}_1 [21, 9.9] and \mathcal{D} [50, Proposition 4.4.10].
- (iv) The following ideals satisfy $\mathcal{I}^{dual} \subseteq \mathcal{I}$: \mathcal{S} [50, Proposition 4.4.8] and \mathcal{DP} [25, 1.15].

Our next aim is to show that the implication E' has \mathcal{I} -AP $\implies E$ has \mathcal{I} -AP holds in some situations not covered by Corollary 4.3. A couple of concepts defined in [15] are needed:

Definition 4.5. Let E be a Banach space. Consider in $\mathcal{L}(E; E)$ the topology, called ν , for which a net (T_α) in $\mathcal{L}(E; E)$ converges to $T \in \mathcal{L}(E; E)$ if and only if

$$\sum_{n=1}^{\infty} x'_n(T_\alpha(x_n)) \longrightarrow \sum_{n=1}^{\infty} x'_n(T(x_n))$$

for every $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ satisfying $\sum_{n=1}^{\infty} \|x'_n\| \|x_n\| < \infty$. In this case we write $T_\alpha \xrightarrow{\nu} T$. It is immediate that the τ_c -topology is stronger than the ν -topology on $\mathcal{L}(E; E)$.

The *weak*-topology* on $\mathcal{L}(E'; E')$ is the topology for which a net (T_α) in $\mathcal{L}(E'; E')$ converges to $T \in \mathcal{L}(E'; E')$ if and only if

$$\sum_{n=1}^{\infty} (T_\alpha(x'_n))x_n \longrightarrow \sum_{n=1}^{\infty} (T(x'_n))x_n$$

for every $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ satisfying $\sum_{n=1}^{\infty} \|x'_n\| \|x_n\| < \infty$. In this case we write $T_\alpha \xrightarrow{weak^*} T$.

The topology ν is stronger than the *weak*-topology* on $\mathcal{L}(E'; E')$. Moreover, for T and a net (T_α) in $\mathcal{L}(E; E)$,

$$T_\alpha \xrightarrow{\nu} T \iff T'_\alpha \xrightarrow{weak^*} T'. \quad (1)$$

Given a Banach space E , by w^* we mean the ordinary weak* topology on E' . For a given operator ideal \mathcal{I} , by $\mathcal{I}_{w^*}(E'; E')$ we denote the set of all

operators belonging to $\mathcal{I}(E'; E')$ which are w^* -to- w^* continuous. The dual space E' is said to have the *weak* density* for \mathcal{I} (in short, E has \mathcal{I} -W*D) if

$$\mathcal{I}(E'; E') \subseteq \overline{\mathcal{I}_{w^*}(E'; E')}^{weak*}.$$

Example 4.6. There are nonreflexive dual Banach spaces having \mathcal{I} -W*D for every operator ideal \mathcal{I} . In [15, Proposition 2.7(a)] it is proved that ℓ_1 has \mathcal{K} -W*D. The only feature of compact operators used in the proof is the ideal property, so the same lines prove that $\ell_1 = (c_0)'$ is a nonreflexive dual Banach space having \mathcal{I} -W*D for every operator ideal \mathcal{I} .

So, formally Corollary 4.3 does not apply to dual spaces having \mathcal{I} -W*D. In this direction we have:

Proposition 4.7. *Let E be a Banach space and let \mathcal{I} be an operator ideal such that $\mathcal{I}^{dual} \subseteq \mathcal{I}$. If E' has \mathcal{I} -AP and \mathcal{I} -W*D, then E has \mathcal{I} -AP.*

Proof. Let $(x_n) \subseteq E$ and $(x'_n) \subseteq E'$ be sequences such that $\sum_{n=1}^{\infty} \|x'_n\| \|x_n\| < \infty$ and $\sum_{n=1}^{\infty} x'_n(T(x_n)) = 0$ for every $T \in \mathcal{I}(E; E)$. We know that $id_{E'} \in \overline{\mathcal{I}(E'; E')^{rc}}$ because E' has \mathcal{I} -AP, and that $\mathcal{I}(E'; E') \subseteq \overline{\mathcal{I}_{w^*}(E'; E')}^{weak*}$ because E has \mathcal{I} -W*D. Thus $id_{E'} \in \overline{\mathcal{I}_{w^*}(E'; E')}^{weak*}$ and then there is a net $(S_\alpha) \subseteq \mathcal{I}_{w^*}(E'; E')$ such that $S_\alpha \xrightarrow{weak*} id_{E'}$. For each α , since S_α is w^* -to- w^* continuous, there is $T_\alpha \in \mathcal{L}(E; E)$ such that $T'_\alpha = S_\alpha$ (see [31, Ex. 3.20]). We know that $S_\alpha \in \mathcal{I}(E'; E')$, so the condition $\mathcal{I}^{dual} \subseteq \mathcal{I}$ yields that $T_\alpha \in \mathcal{I}(E; E)$ for every α . From (1) and

$$T'_\alpha = S_\alpha \xrightarrow{weak*} id_{E'} = (id_E)'$$

we get $T_\alpha \xrightarrow{\nu} id_E$. So, by the definition of the ν -convergence,

$$\sum_{n=1}^{\infty} x'_n(T_\alpha(x_n)) \longrightarrow \sum_{n=1}^{\infty} x'_n(id_E(x_n)) = \sum_{n=1}^{\infty} x'_n(x_n).$$

But $\sum_{n=1}^{\infty} x'_n(T_\alpha(x_n)) = 0$ for every α because each $T_\alpha \in \mathcal{I}(E; E)$, therefore $\sum_{n=1}^{\infty} x'_n(x_n) = 0$. By Proposition 3.4 it follows that E has \mathcal{I} -AP. \square

5 Tensor stability

In this section we study the stability of \mathcal{I} -AP under the formation of projective tensor products. By $E_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E_n$ we mean the completed projective tensor product of E_1, \dots, E_n ($\hat{\otimes}_\pi^n E$ if $E = E_1 = \cdots = E_n$), and by $\hat{\otimes}_\pi^{n,s} E$ the completed n -fold symmetric projective tensor product of E .

Definition 5.1. Given continuous linear operators $u_j: E_j \rightarrow F_j, j = 1, \dots, n$, by $u_1 \otimes \dots \otimes u_n$ we denote the (unique) continuous linear operator from $E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ to $F_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi F_n$ such that

$$u_1 \otimes \dots \otimes u_n(x_1 \otimes \dots \otimes x_n) = u_1(x_1) \otimes \dots \otimes u_n(x_n)$$

for every $x_1 \in E_1, \dots, x_n \in E_n$.

The proof of the stability of the approximation property with respect to the formation of projective tensor products relies heavily on the fact that $u_1 \otimes \dots \otimes u_n$ is a finite rank operator whenever u_1, \dots, u_n are finite rank operators. Let us see that this does not hold for arbitrary operator ideals:

Example 5.2. The identity operator id_{ℓ_2} is weakly compact but $id_{\ell_2} \otimes id_{\ell_2} = id_{\ell_2 \hat{\otimes}_\pi \ell_2}$ is not weakly compact because $\ell_2 \hat{\otimes}_\pi \ell_2$ fails to be reflexive.

In order to settle this difficulty we need the following methods of generating ideals of multilinear mappings from operator ideals. By $\mathcal{L}(E_1, \dots, E_n; F)$ we denote the space of continuous n -linear mappings from $E_1 \times \dots \times E_n$ to F endowed with the usual sup norm.

Definition 5.3. Let $\mathcal{I}, \mathcal{I}_1, \dots, \mathcal{I}_n$ be operator ideals.

(a) (Factorization Method) A continuous n -linear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ is said to be of type $\mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n]$ if there are Banach spaces G_1, \dots, G_n , linear operators $u_j \in \mathcal{I}_j(E_j; G_j), j = 1, \dots, n$, and an n -linear mapping $B \in \mathcal{L}(G_1, \dots, G_n; F)$ such that $A = B \circ (u_1, \dots, u_n)$. In this case we write $A \in \mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; F)$. If $\mathcal{I} = \mathcal{I}_1 = \dots = \mathcal{I}_n$ we simply write $\mathcal{L}[\mathcal{I}]$.

(b) (Composition ideals) A continuous n -linear mapping $A \in \mathcal{L}(E_1, \dots, E_n; F)$ belongs to $\mathcal{I} \circ \mathcal{P}$ if there are a Banach space G , an n -linear mapping $B \in \mathcal{L}(E_1, \dots, E_n; G)$ and a linear operator $u \in \mathcal{I}(G; F)$ such that $A = u \circ B$. In this case we write $A \in \mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; F)$.

For details and examples we refer to [6, 7].

Proposition 5.4. Let $\mathcal{I}, \mathcal{I}_1, \dots, \mathcal{I}_n$ be operator ideals such that $\mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n] \subseteq \mathcal{I} \circ \mathcal{L}$. If E_1 has \mathcal{I}_1 -AP, ..., E_n has \mathcal{I}_n -AP, then $E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ has \mathcal{I} -AP.

Proof. Let K be a compact subset of $E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$. By [21, Corollary 3.5.1] there are compact sets $K_1 \subseteq E_1, \dots, K_n \subseteq E_n$ such that K is contained in the closure of the absolutely convex hull of $K_1 \otimes \dots \otimes K_n := \{x_1 \otimes \dots \otimes x_n : x_1 \in K_1, \dots, x_n \in K_n\}$. Since compact sets are bounded there is $M > 0$ such that $\|x_j\| \leq M$ for every $x_j \in E_j, j = 1, \dots, n$. Let $\varepsilon > 0$. As E_1 has \mathcal{I}_1 -AP, there is an operator $u_1 \in \mathcal{I}_1(E_1; E_1)$ such that $\|u_1(x_1) - x_1\| < \frac{\varepsilon}{2nM^{n-1}}$ for every $x_1 \in K_1$. As E_2 has \mathcal{I}_2 -AP, there is an operator $u_2 \in \mathcal{I}_2(E_2; E_2)$ such that $\|u_2(x_2) - x_2\| < \frac{\varepsilon}{2nM^{n-1}\|u_1\|}$ for every $x_2 \in K_2$. Repeating the procedure we obtain operators $u_j \in \mathcal{I}_j(E_j; E_j)$ such that

$$\|u_j(x_j) - x_j\| < \frac{\varepsilon}{2nM^{n-1}\|u_1\| \dots \|u_{j-1}\|}$$

for every $x_j \in K_j$, $j = 1, \dots, n$. Consider the canonical n -linear mapping $\sigma_n: E_1 \otimes \dots \otimes E_n \longrightarrow E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ given by $\sigma_n(x_1, \dots, x_n) = x_1 \otimes \dots \otimes x_n$ and observe that $\sigma_n \circ (u_1, \dots, u_n) \in \mathcal{L}[\mathcal{I}_1, \dots, \mathcal{I}_n](E_1, \dots, E_n; E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n)$. By assumption we have $\sigma_n \circ (u_1, \dots, u_n) \in \mathcal{I} \circ \mathcal{L}(E_1, \dots, E_n; E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n)$. Calling T the linearization of $\sigma_n \circ (u_1, \dots, u_n)$, by [7, Proposition 3.2(a)] we have that $T \in \mathcal{I}(E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n)$. For every $x_1 \in E_1, \dots, x_n \in E_n$,

$$\begin{aligned} T(x_1 \otimes \dots \otimes x_n) &= \sigma_n \circ (u_1, \dots, u_n)(x_1, \dots, x_n) \\ &= \sigma_n(u_1(x_1), \dots, u_n(x_n)) \\ &= u_1(x_1) \otimes \dots \otimes u_n(x_n) \\ &= u_1 \otimes \dots \otimes u_n(x_1 \otimes \dots \otimes x_n). \end{aligned}$$

As both T and $u_1 \otimes \dots \otimes u_n$ are linear it follows that $T = u_1 \otimes \dots \otimes u_n$, hence $u_1 \otimes \dots \otimes u_n \in \mathcal{I}(E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n; E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n)$. We shall denote the projective norm of a tensor $z \in E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ by $\|z\|$ instead of $\pi(z)$. Given $x_1 \in K_1, \dots, x_n \in K_n$,

$$\begin{aligned} &\|u_1 \otimes \dots \otimes u_n(x_1 \otimes \dots \otimes x_n) - x_1 \otimes \dots \otimes x_n\| = \|u_1(x_1) \otimes \dots \otimes u_n(x_n) - x_1 \otimes \dots \otimes x_n\| \\ &= \|u_1(x_1) \otimes \dots \otimes u_n(x_n) - \sum_{j=1}^{n-1} u_1(x_1) \otimes \dots \otimes u_j(x_j) \otimes x_{j+1} \otimes \dots \otimes x_n \\ &\quad + \sum_{j=1}^{n-1} u_1(x_1) \otimes \dots \otimes u_j(x_j) \otimes x_{j+1} \otimes \dots \otimes x_n - x_1 \otimes \dots \otimes x_n\| \\ &= \left\| \sum_{j=1}^n u_1(x_1) \otimes \dots \otimes u_{j-1}(x_{j-1}) \otimes (u_j(x_j) - x_j) \otimes x_{j+1} \otimes \dots \otimes x_n \right\| \\ &\leq \sum_{j=1}^n \|u_1(x_1) \otimes \dots \otimes u_{j-1}(x_{j-1}) \otimes (u_j(x_j) - x_j) \otimes x_{j+1} \otimes \dots \otimes x_n\| \\ &\leq \sum_{j=1}^n \|u_1\| \|x_1\| \dots \|u_{j-1}\| \|x_{j-1}\| \|u_j(x_j) - x_j\| \|x_{j+1}\| \dots \|x_n\| \\ &< \sum_{j=1}^n \|u_1\| \dots \|u_{j-1}\| M^{n-1} \frac{\varepsilon}{2nM^{n-1} \|u_1\| \dots \|u_{j-1}\|} = \frac{\varepsilon}{2}. \end{aligned}$$

In summary,

$$\|u_1 \otimes \dots \otimes u_n(x_1 \otimes \dots \otimes x_n) - x_1 \otimes \dots \otimes x_n\| < \frac{\varepsilon}{2},$$

for every $x_1 \in K_1, \dots, x_n \in K_n$. Take z in the absolutely convex hull of $K_1 \otimes \dots \otimes K_n$. Then $z = \sum_{j=1}^k \lambda_j x_j^1 \otimes \dots \otimes x_j^n$, where $k \in \mathbb{N}$, $\lambda_1, \dots, \lambda_k$ are

scalars with $|\lambda_1| + \dots + |\lambda_k| \leq 1$, and $x_j^m \in K_m$ for $j = 1, \dots, k, m = 1, \dots, n$. Then

$$\begin{aligned}
& \|u_1 \otimes \dots \otimes u_n(z) - z\| \\
&= \left\| u_1 \otimes \dots \otimes u_n \left(\sum_{j=1}^k \lambda_j x_j^1 \otimes \dots \otimes x_j^n \right) - \sum_{j=1}^k \lambda_j x_j^1 \otimes \dots \otimes x_j^n \right\| \\
&= \left\| \sum_{j=1}^k \lambda_j (u_1 \otimes \dots \otimes u_n (x_j^1 \otimes \dots \otimes x_j^n) - x_j^1 \otimes \dots \otimes x_j^n) \right\| \\
&\leq \sum_{j=1}^k |\lambda_j| \|(u_1 \otimes \dots \otimes u_n (x_j^1 \otimes \dots \otimes x_j^n) - x_j^1 \otimes \dots \otimes x_j^n)\| \\
&< \frac{\varepsilon}{2} \sum_{j=1}^k |\lambda_j| = \frac{\varepsilon}{2}.
\end{aligned}$$

By continuity we have that

$$\|u_1 \otimes \dots \otimes u_n(z) - z\| \leq \frac{\varepsilon}{2} < \varepsilon$$

for every z in the closure of the absolutely convex hull of $K_1 \otimes \dots \otimes K_n$, hence for every $z \in K$. \square

As to ideals satisfying the conditions above we have:

- Example 5.5.** (a) It is plain that $\mathcal{L}[\mathcal{F}] \subseteq \mathcal{F} \circ \mathcal{L}$ and $\mathcal{L}[\mathcal{S}] \subseteq \mathcal{S} \circ \mathcal{L}$.
(b) $\mathcal{L}[\mathcal{N}_1] \subseteq \mathcal{N}_1 \circ \mathcal{L}$ [33, Theorem 3.7] (see also [35, Proposition 17.3.9]).
(c) $\mathcal{L}[\mathcal{J}] \subseteq \mathcal{J} \circ \mathcal{L}$ [34, Theorem 2].
(d) Let $\mathcal{L}_{\mathcal{K}}$ denote the ideal of compact multilinear mappings (bounded sets are sent to relatively compact sets). Pełczyński [49] proved that $\mathcal{K} \circ \mathcal{L} = \mathcal{L}_{\mathcal{K}}$. Now it follows easily that $\mathcal{L}[\mathcal{K}] \subseteq \mathcal{K} \circ \mathcal{L}$. So the projective tensor product of spaces with CAP has CAP too.
(e) $\mathcal{L}[\mathcal{L}_{\infty, q, \gamma}] \subseteq \mathcal{L}_{\infty, q, \gamma} \circ \mathcal{L}$ for $0 < q \leq 1$ and $-1/q < \gamma < \infty$ [17, Theorem 3.1].
(f) $\mathcal{L}[\mathcal{L}_{1, q}] \subseteq \mathcal{L}_{1, q} \circ \mathcal{L}$ for $q > 1$ and $\mathcal{L}[\mathcal{K}_{1, p}] \subseteq \mathcal{K}_{1, p} \circ \mathcal{L}$ for $p \geq 1$ [12, Theorem 2.1].
(g) It is unknown if the projective tensor product of Schur spaces is a Schur space (see, e.g., [8]), so it is unknown if $\mathcal{L}[\mathcal{CC}] \subseteq \mathcal{CC} \circ \mathcal{L}$.

Here are other concrete situations to which Proposition 5.4 applies:

Lemma 5.6. *Let $n \in \mathbb{N}$.*

- (a) *If $1 \leq p_1, \dots, p_n < \infty$, then $\mathcal{L}[\mathcal{W}, \mathcal{I}_1, \dots, \mathcal{I}_n] \subseteq \mathcal{W} \circ \mathcal{L}$ where \mathcal{I}_j is either \mathcal{K} or Π_{p_j} , $j = 1, \dots, n$.*
(b) $\mathcal{L}[\Pi_1, \mathcal{J}, \overset{(n)}{\cdot}, \mathcal{J}] \subseteq \Pi_1 \circ \mathcal{L}$.
(c) $\mathcal{L}[\mathcal{QN}, \mathcal{N}_1, \overset{(n)}{\cdot}, \mathcal{N}_1] \subseteq \mathcal{QN} \circ \mathcal{L}$.
(d) *If $p_1 > p_j$ for $j = 2, \dots, n$, then $\mathcal{L}[\mathcal{U}_{p_1}, \mathcal{U}_{p_2}, \dots, \mathcal{U}_{p_n}] \subseteq \mathcal{U}_{p_1} \circ \mathcal{L}$.*

Proof. (a) Given an n -linear mapping $A \in \mathcal{L}[\mathcal{W}, \mathcal{I}_1, \dots, \mathcal{I}_n](E, E_1, \dots, E_n; F)$, write $A = B \circ (u, u_1, \dots, u_n)$ with $u \in \mathcal{W}(E; G)$, $u_j \in \mathcal{I}_j(E_j; G_j)$, $j = 1, \dots, n$, and $B \in \mathcal{L}(G, G_1, \dots, G_n; F)$. Since u is weakly compact and u_1 is either compact or absolutely p_1 -summing, by a result of Racher [52] we have that $u \otimes u_1$ is weakly compact. As u_2 is either compact or absolutely p_2 -summing and the projective tensor norm is associative, $u \otimes u_1 \otimes u_2 = (u \otimes u_1) \otimes u_2$ is weakly compact by the same result of [52]. Repeating this procedure finitely many times we conclude that $u \otimes u_1 \otimes \dots \otimes u_n$ is weakly compact. Denoting by $\sigma_{n+1}: E \times E_1 \times \dots \times E_n \rightarrow E \hat{\otimes}_\pi E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ the canonical $(n+1)$ -linear mapping and by T the linearization of B , we conclude that $A = T \circ (u \otimes u_1 \otimes \dots \otimes u_n) \circ \sigma_{n+1}$. It follows that $A \in \mathcal{W} \circ \mathcal{L}(E, E_1, \dots, E_n; F)$ because $T \circ (u \otimes u_1 \otimes \dots \otimes u_n)$ is weakly compact by the ideal property.

For (b), (c) and (d), repeat the proof above using, instead of Racher's result, the following results: in (b) and (c), two results due to Holub [34]: (i) If $u_1 \in \Pi_1(E_1; F_1)$ and $u_2 \in \mathcal{J}(E_2; F_2)$ then $u_1 \otimes u_2 \in \Pi_1(E_1 \hat{\otimes}_\pi E_2; F_1 \hat{\otimes}_\pi F_2)$; (ii) If $u_1 \in \mathcal{QN}(E_1; F_1)$ and $u_2 \in \mathcal{N}_1(E_2; F_2)$ then $u_1 \otimes u_2 \in \mathcal{QN}(E_1 \hat{\otimes}_\pi E_2; F_1 \hat{\otimes}_\pi F_2)$. In (d), the following result, which appears in König [36, p. 79] and is credited to Pietsch [51]: if $u_1 \in \mathcal{U}_{p_1}(E_1; F_1)$, $u_2 \in \mathcal{U}_{p_2}(E_2; F_2)$ and $p_1 > p_2$, then $u_1 \otimes u_2 \in \mathcal{U}_{p_1}(E_1 \hat{\otimes}_\pi E_2; F_1 \hat{\otimes}_\pi F_2)$. \square

Combining Proposition 5.4 and Lemma 5.6 we get:

Proposition 5.7.

- (a) Let E_1, \dots, E_n be Banach spaces, one of which with WCAP and the others E_j with either CAP or Π_{p_j} -AP for some $1 \leq p_j < \infty$. Then $E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ has WCAP.
- (b) Let E_1, \dots, E_n be Banach spaces, one of which with Π_1 -AP and the others with \mathcal{J} -AP. Then $E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ has Π_1 -AP.
- (c) Let E_1, \dots, E_n be Banach spaces, one of which with \mathcal{QN} -AP and the others with AP. Then $E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ has AP.
- (d) Let $0 < p_1, \dots, p_n$. If E_1, \dots, E_n are Banach spaces, each E_j with \mathcal{U}_{p_j} -AP, then $E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ has \mathcal{U}_{p_k} -AP if $p_k > p_j$ for every $j \neq k$.

Example 5.8. Let E be a Banach space with WCAP but not with CAP (see the Introduction), and E_1, \dots, E_n be Banach spaces such that each E_j has either CAP or Π_{p_j} -AP, $1 \leq p_1, \dots, p_n < \infty$. Then $E \hat{\otimes}_\pi E_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi E_n$ has WCAP.

Corollary 5.9. Let \mathcal{I} be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$. The following are equivalent for a Banach space E :

- (a) E has \mathcal{I} -AP.
- (b) $\hat{\otimes}_\pi^n E$ has \mathcal{I} -AP for every $n \in \mathbb{N}$.
- (c) $\hat{\otimes}_\pi^n E$ has \mathcal{I} -AP for some $n \in \mathbb{N}$.
- (d) $\hat{\otimes}_\pi^{n,s} E$ has \mathcal{I} -AP for every $n \in \mathbb{N}$.
- (e) $\hat{\otimes}_\pi^{n,s} E$ has \mathcal{I} -AP for some $n \in \mathbb{N}$.

Proof. (a) \implies (b) follows from Proposition 5.4; (b) \implies (c) is obvious; (c) \implies (a) follows from Proposition 3.5 because E is obviously a complemented subspace of $\hat{\otimes}_\pi^n E$; (b) \implies (d) follows from Proposition 3.5 because $\hat{\otimes}_\pi^{n,s} E$ a complemented subspace of $\hat{\otimes}_\pi^n E$ via the symmetrization operator; (d) \implies (e) is obvious; (e) \implies (a) follows from Proposition 3.5 because E is a complemented subspace of $\hat{\otimes}_\pi^{n,s} E$ (see [5, Corollary 4]). \square

6 Polynomial ideals and the \mathcal{I} -AP

The symbol $\mathcal{P}(^n E; F)$ stands for the space of continuous n -homogeneous polynomials from E to F . A *polynomial ideal* is a subclass \mathcal{Q} of the class of all continuous homogeneous polynomials between Banach spaces such that, for every $n \in \mathbb{N}$ and Banach spaces E and F , the component $\mathcal{Q}(^n E; F) := \mathcal{P}(^n E; F) \cap \mathcal{Q}$ satisfy:

- (a) $\mathcal{Q}(^n E; F)$ is a linear subspace of $\mathcal{P}(^n E; F)$ which contains the n -homogeneous polynomials of finite type,
- (b) If $T \in \mathcal{L}(G; E)$, $P \in \mathcal{Q}(^n E; F)$ and $S \in \mathcal{L}(F; H)$, then $S \circ P \circ T \in \mathcal{Q}(^n G; H)$.

There are different ways to construct a polynomial ideal from a given operator ideal \mathcal{I} . Let us see three of such methods (see [6, 7]):

Definition 6.1. Let \mathcal{I} be an operator ideal.

- (a) (Factorization Method) A continuous n -homogeneous polynomial $P \in \mathcal{P}(^n E; F)$ is said to be of type $\mathcal{L}[\mathcal{I}]$ if there are a Banach space G , a linear operator $u \in \mathcal{I}(E; G)$ and a polynomial $Q \in \mathcal{P}(^n G; F)$ such that $P = Q \circ u$. In this case we write $P \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)$
- (b) (Composition ideals) A continuous n -homogeneous polynomial $P \in \mathcal{P}(^n E; F)$ belongs to $\mathcal{I} \circ \mathcal{P}$ if there are a Banach space G , a polynomial $Q \in \mathcal{P}(^n G; F)$ and a linear operator $u \in \mathcal{I}(E; G)$ such that $P = u \circ Q$. In this case we write $P \in \mathcal{I} \circ \mathcal{P}(^n E; F)$.
- (c) (Linearization method) A continuous n -homogeneous polynomial $P \in \mathcal{P}(^n E; F)$ is said to be of type $[\mathcal{I}]$ if the linear operator

$$\bar{P} : E \rightarrow \mathcal{P}(^{n-1} E; F), \quad \bar{P}(x)(y) = \check{P}(x, y, \dots, y)$$

belongs to \mathcal{I} . In this case we write $P \in \mathcal{P}_{[\mathcal{I}]}(^n E; F)$.

It is well known that $\mathcal{L}[\mathcal{I}]$, $\mathcal{I} \circ \mathcal{P}$ and $[\mathcal{I}]$ are polynomial ideals.

Given a polynomial $P \in \mathcal{P}(^n E; F)$, by \check{P} we mean the (unique) continuous symmetric n -linear mapping from E^n to F such that $P(x) = \check{P}(x, \dots, x)$ for every $x \in E$.

Theorem 6.2. Let \mathcal{I} be an operator ideal. The following are equivalent for a Banach space E :

- (a) E has the \mathcal{I} -approximation property.
- (b) $\mathcal{P}(^n E; F) = \overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)}^{\tau_c}$ for every $n \in \mathbb{N}$ and every Banach space F .

- (c) $\mathcal{P}(^n E; F) = \overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)}^{\tau_c}$ for some $n \in \mathbb{N}$ and every Banach space F .
(d) $\mathcal{P}(^n F; E) = \overline{\mathcal{I} \circ \mathcal{P}(^n F; E)}^{\tau_c}$ for every $n \in \mathbb{N}$ and every Banach space F .
(e) $\mathcal{P}(^n F; E) = \overline{\mathcal{I} \circ \mathcal{P}(^n F; E)}^{\tau_c}$ for some $n \in \mathbb{N}$ and every Banach space F .

Furthermore, if $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$, then the conditions above are also equivalent to:

- (f) $\mathcal{P}(^n E; F) = \overline{\mathcal{I} \circ \mathcal{P}(^n E; F)}^{\tau_c}$ for every $n \in \mathbb{N}$ and every Banach space F .
(g) $\mathcal{P}(^n E; F) = \overline{\mathcal{I} \circ \mathcal{P}(^n E; F)}^{\tau_c}$ for some $n \in \mathbb{N}$ and every Banach space F .

Proof. (a) \implies (b) Let $P \in \mathcal{P}(^n E; F)$, K be a compact subset of E and $\varepsilon > 0$. Since P is uniformly continuous on K , there is $\delta > 0$ such that $\|P(y) - P(x)\| < \varepsilon$ whenever $\|y - x\| < \delta$, $x \in K$ and $y \in E$. By the \mathcal{I} -AP of E there is an operator $T \in \mathcal{I}(E; E)$ such that $\|T(x) - x\| < \delta$ for every $x \in K$. It follows that $\|P(T(x)) - P(x)\| < \varepsilon$ for every $x \in K$. But $P \circ T \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)$, so we have that $P \in \overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)}^{\tau_c}$.

(c) \implies (a) Let $u \in \mathcal{L}(E; F)$, K be a compact subset of E and $\varepsilon > 0$. Let $\varphi \in E'$, $\varphi \neq 0$, and $a \in K$ be such that $\varphi(a) = 1$. Define $P \in \mathcal{P}(^n E; F)$ by $P(x) = \varphi(x)^{n-1}u(x)$. Since $K_1 := \bigcup_{\varepsilon_i = \pm 1} (\varepsilon_1 K + \varepsilon_2 K + \dots + \varepsilon_n K)$ is a compact subset of E , by assumption there is a polynomial $Q \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)$ such that $\|Q(x) - P(x)\| < \frac{n! \varepsilon}{n}$ for every $x \in K_1$. By the polarization formula, for every $(x_1, x_2, \dots, x_n) \in K \times K \dots \times K$ we have

$$\begin{aligned} & \|\check{Q}(x_1, x_2, \dots, x_n) - \check{P}(x_1, x_2, \dots, x_n)\| \\ &= \left\| \frac{1}{n! 2^n} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \varepsilon_2 \dots \varepsilon_n \left[Q \left(\sum_{i=1}^n \varepsilon_i x_i \right) - P \left(\sum_{i=1}^n \varepsilon_i x_i \right) \right] \right\| < \frac{\varepsilon}{n}. \end{aligned}$$

From

$$\check{P}(x, a, \dots, a) = \frac{1}{n} u(x) + \frac{(n-1)}{n} \varphi(x) u(a),$$

it results that

$$\begin{aligned} & \|n\check{Q}(x, a, \dots, a) - u(x) - (n-1)\varphi(x)u(a)\| \\ &= n \left\| \check{Q}(x, a, \dots, a) - \left(\frac{1}{n} u(x) + \frac{(n-1)}{n} \varphi(x) u(a) \right) \right\| < \varepsilon \end{aligned}$$

for every $x \in K$. Considering $S = n\check{Q}(\cdot, a, \dots, a) - (n-1)\varphi(\cdot)u(a) \in \mathcal{L}(E; F)$, we have $\|S(x) - u(x)\| < \varepsilon$ for every $x \in K$. Let us check that $S \in \mathcal{I}(E; F)$. Indeed, as $Q \in \mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)$, there are a Banach space G , an operator $v \in \mathcal{I}(E; G)$ and a polynomial $R \in \mathcal{P}(^n G; F)$ such that $Q = R \circ v$. Define $T: G \rightarrow F$ by $T(y) = \check{R}(y, v(a), \dots, v(a))$. Then $T \circ v \in \mathcal{I}(E; F)$ and

$$T \circ v(x) = T(v(x)) = \check{R}(v(x), v(a), \dots, v(a)) = \check{Q}(x, a, \dots, a),$$

for every $x \in E$, proving that $\check{Q}(\cdot, a, \dots, a) \in \mathcal{I}(E; F)$. On the other hand, the operator $\varphi(\cdot)u(a) \in \mathcal{I}(E; F)$ as it is a finite rank operator. Thus $S \in \mathcal{I}(E; F)$

and $\mathcal{L}(E; F) = \overline{\mathcal{I}(E; F)}^{\tau_c}$. Calling on Proposition 3.4 we have that E has \mathcal{I} -AP.

(a) \implies (d) Let $P \in \mathcal{P}(^n F; E)$, K be a compact subset of E and $\varepsilon > 0$. Since $P(K)$ is a compact subset of E and E has the \mathcal{I} -approximation property, there is an operator $T \in \mathcal{I}(E; E)$ such that $\|T(z) - z\| < \varepsilon$ for every $z \in P(K)$. Hence $\|T(P(x)) - P(x)\| < \varepsilon$ for every $x \in K$. Since $T \circ P \in \mathcal{I} \circ \mathcal{P}(^n F; E)$ we have that $P \in \overline{\mathcal{I} \circ \mathcal{P}(^n F; E)}^{\tau_c}$.

(e) \implies (a) The same argument of (c) \implies (a), *mutatis mutandis*, works in this case. We just sketch the proof: given an operator $u \in \mathcal{L}(F; E)$, a compact set $K \subseteq F$ and $\varepsilon > 0$, take $\varphi \in F'$, $\varphi \neq 0$, and $a \in K$ such that $\varphi(a) = 1$. Defining $P = \varphi(\cdot)^{n-1} u(\cdot) \in \mathcal{P}(^n F; E)$ and a compact subset K_1 of F as before, by assumption there is a polynomial $Q \in \mathcal{I} \circ \mathcal{P}(^n F; E)$ such that $\|Q(x) - P(x)\| < \frac{n! \varepsilon}{n}$ for every $x \in K_1$. Define $S = n\check{Q}(\cdot, a, \dots, a) - (n-1)\varphi(\cdot)u(a) \in \mathcal{L}(F; E)$ and proceed exactly as above to get $\|S(x) - u(x)\| < \varepsilon$ for every $x \in K$. Write $Q = v \circ R$ with $v \in \mathcal{I}(G; E)$ and $R \in \mathcal{P}(^n F; G)$ and define $T \in \mathcal{L}(F; G)$ by $T(y) = \check{R}(y, a, \dots, a)$. Thus $v \circ T = \check{Q}(\cdot, a, \dots, a) \in \mathcal{I}(F; E)$ and this implies that $S \in \mathcal{I}(F; E)$.

Since (b) \implies (c) and (d) \implies (e) are obvious and (b) \implies (a) and (c) \implies (a) follow by taking $n = 1$ in Proposition 3.4, the first part of the proof is complete.

Assume now that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$.

(a) \implies (f) E has \mathcal{I} -AP by assumption. Let $n \in \mathbb{N}$, $P \in \mathcal{P}(^n E; F)$, K be a compact subset of E and $\varepsilon > 0$. Note that $P = P_L \circ \sigma_n$ where $\sigma_n \in \mathcal{P}(^n E; \hat{\otimes}_\pi^{n,s} E)$ is the canonical n -homogeneous polynomial defined by $\sigma_n(x) = x \otimes \dots \otimes x$ and $P_L \in \mathcal{L}(\hat{\otimes}_\pi^{n,s} E; F)$ is the linearization of P , that is $P_L(x \otimes \dots \otimes x) = P(x)$. By Corollary 5.9 we have that $\hat{\otimes}_\pi^{n,s} E$ has \mathcal{I} -AP, hence $\mathcal{L}(\hat{\otimes}_\pi^{n,s} E; F) = \overline{\mathcal{I}(\hat{\otimes}_\pi^{n,s} E; F)}^{\tau_c}$ by Proposition 3.4. So for the compact subset $\sigma(K)$ of $\hat{\otimes}_\pi^{n,s} E$ there is an operator $u \in \mathcal{I}(\hat{\otimes}_\pi^{n,s} E; F)$ such that

$$\|u \circ \sigma_n(x) - P(x)\| = \|u(\sigma_n(x)) - P_L(\sigma_n(x))\| < \varepsilon$$

for every $x \in K$. Since $Q = u \circ \sigma_n \in \mathcal{I} \circ \mathcal{P}(^n E; F)$ we have that $P \in \overline{\mathcal{I} \circ \mathcal{P}(^n E; F)}^{\tau_c}$.

(f) \implies (g) is obvious and (g) \implies (a) follows from a repetition of the argument of the proofs of (c) \implies (a) and (e) \implies (a), therefore the proof is complete. \square

The spaces $\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; E)$ and $\mathcal{I} \circ \mathcal{P}(^n E; E)$ are often different. We have obtained situations where, even though different, their τ_c -closures coincide:

Corollary 6.3. *Let \mathcal{I} be an operator ideal.*

(a) *If the Banach spaces E and F have the \mathcal{I} -approximation property, then $\overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; F)}^{\tau_c} = \mathcal{P}(^n E; F) = \overline{\mathcal{I} \circ \mathcal{P}(^n F; E)}^{\tau_c}$ for every $n \in \mathbb{N}$.*

(b) *A Banach space E has the \mathcal{I} -approximation property if and only if $\overline{\mathcal{P}_{\mathcal{L}[\mathcal{I}]}(^n E; E)}^{\tau_c} = \mathcal{P}(^n E; E) = \overline{\mathcal{I} \circ \mathcal{P}(^n E; E)}^{\tau_c}$ for every $n \in \mathbb{N}$.*

Example 6.4. It is not difficult to check that neither $\mathcal{P}_{\mathcal{L}[\mathcal{W}]}(^2\ell_1; \ell_1) \subseteq \mathcal{W} \circ \mathcal{P}(^2\ell_1; \ell_1)$ nor $\mathcal{W} \circ \mathcal{P}(^2\ell_1; \ell_1) \subseteq \mathcal{P}_{\mathcal{L}[\mathcal{W}]}(^2\ell_1; \ell_1)$ (see [6, Examples 27 and 28]). Nevertheless, by Corollary 6.3(b) both subspaces are τ_c -dense in $\mathcal{P}(^2\ell_1; \ell_1)$ because ℓ_1 has the approximation property (hence has the weakly compact approximation property).

The following result appears in [20]:

Theorem 6.5. [20, Theorem 11] *The following are equivalent for a Banach space E :*

- (a) *E has the weakly compact approximation property.*
- (b) $\mathcal{P}(^n E; F) = \overline{\mathcal{P}_{[\mathcal{W}]}(^n E; F)}^{\tau_c}$ for every $n \in \mathbb{N}$ and every Banach space F .
- (c) $\mathcal{P}(^n E; F) = \overline{\mathcal{P}_{[\mathcal{W}]}(^n E; F)}^{\tau_c}$ for some $n \in \mathbb{N}$ and every Banach space F .

Unfortunately there is a gap in the proof of this theorem (see the MathSciNet review of this paper by C. Boyd). In this direction we have:

Proposition 6.6. *Let \mathcal{I} be a closed injective operator ideal. The following are equivalent for a Banach space E :*

- (a) *E has the \mathcal{I} -approximation property.*
- (b) $\mathcal{P}(^n E; F) = \overline{\mathcal{P}_{[\mathcal{I}]}(^n E; F)}^{\tau_c}$ for every $n \in \mathbb{N}$ and every Banach space F .
- (c) $\mathcal{P}(^n E; F) = \overline{\mathcal{P}_{[\mathcal{I}]}(^n E; F)}^{\tau_c}$ for some $n \in \mathbb{N}$ and every Banach space F .

Proof. Just combine Theorem 6.2 with the fact that $[\mathcal{I}] = \mathcal{L}[\mathcal{I}]$ whenever the operator ideal \mathcal{I} is closed and injective (see [11]). \square

Recalling that \mathcal{W} is closed and injective, Proposition 6.6 fixes Theorem 6.5 and generalizes it to arbitrary closed injective operator ideals.

7 Spaces of holomorphic functions

The approximation property and its variants in spaces of holomorphic functions and their preduals have been largely investigated (see, e.g., [3, 10, 18, 19, 29, 30, 46]). In this section we study the \mathcal{I} -approximation property in spaces of holomorphic functions of bounded type, spaces of weakly uniformly continuous holomorphic functions, spaces of bounded holomorphic functions and/or their preduals. For background on infinite-dimensional holomorphy we refer to [28, 44]. An important issue of this section is the combination of results from different sections of the paper.

All spaces in this section are supposed to be complex.

Spaces of holomorphic functions, spaces of bounded holomorphic functions and spaces of weakly uniformly continuous holomorphic functions, as well as their corresponding preduals, are locally convex spaces, so we have to say a few words about the definition of the \mathcal{I} -approximation property in the setting of locally convex spaces. The definition of operator ideals (on Banach spaces)

can be naturally generalized to the concept of operator ideals on locally convex spaces (details can be found in [50, Chapter 29]). We say that an operator ideal \mathcal{U} on locally convex spaces is an extension of an operator ideal \mathcal{I} on Banach spaces if $\mathcal{U}(E; F) = \mathcal{I}(E; F)$ for all Banach spaces E and F . There are several ways to extend an operator ideal on Banach spaces to an operator ideal on locally convex spaces (see [50, Section 29.5]). In this paper we shall work with the smallest of such natural extensions, which we describe next. Given an operator ideal \mathcal{I} on Banach spaces, an operator $S \in \mathcal{L}(U; V)$ between locally convex spaces belongs to the *inferior extension of \mathcal{I}* if there exist Banach spaces E and F and operators $A \in \mathcal{L}(U, E)$, $T \in \mathcal{I}(E, F)$ and $Y \in \mathcal{L}(F, V)$ such that $S = Y \circ T \circ A$. In this case, for the sake of simplicity, we still write $S \in \mathcal{I}(U; V)$. Of course we can consider the compact-open topology on $\mathcal{L}(U; U)$ for a locally convex space U , so Definition 1.1 makes sense for an operator ideal \mathcal{I} on Banach spaces and a locally convex space U , hence the \mathcal{I} -approximation property is well defined for locally convex spaces.

Unless explicitly stated otherwise, an operator ideal means an operator ideal on Banach spaces and an statement like $\mathcal{I}_1 \subseteq \mathcal{I}_2$ means that $\mathcal{I}_1(E; F) \subseteq \mathcal{I}_2(E; F)$ for all Banach spaces E and F .

Remark 7.1. It is easy to see that Propositions 3.1, 3.5 and 3.6 hold true in the realm of locally convex spaces. Of course, in condition (d) of Proposition 3.1, $\|T(x) - x\|$ is replaced by $p(T(x) - x)$ where p is an arbitrary continuous semi-norm. The proof of the locally convex version of Proposition 3.1 follows the lines of the proof of [37, 43(1)].

Definition 7.2. A sequence $\{E_n\}_{n=1}^\infty$ of subspaces of a locally convex space E is said to be a *decomposition of E* if any $x \in E$ can be written in a unique way as $x = \sum_{n=1}^\infty x_n$ with $x_n \in E_n$ for every n and the projection $\sum_{n=1}^\infty x_n \mapsto \sum_{n=1}^m x_n$ is continuous for every $m \in \mathbb{N}$.

Let $\mathcal{S} = \{(\alpha_n)_{n=1}^\infty : \alpha_n \in \mathbb{C} \text{ and } \limsup_{n \rightarrow \infty} |\alpha_n|^{\frac{1}{n}} \leq 1\}$. A decomposition $\{E_n\}_{n=1}^\infty$ of E is an *\mathcal{S} -absolute decomposition* if

- (1) $\sum_{n=1}^\infty x_n \in E$, $x_n \in E_n$ for all n and $(\alpha_n)_{n=1}^\infty \in \mathcal{S}$ implies $\sum_{n=1}^\infty \alpha_n x_n \in E$,
- (2) If p is a continuous semi-norm on E and $(\alpha_n)_{n=1}^\infty \in \mathcal{S}$ then

$$p_\alpha \left(\sum_{n=1}^\infty x_n \right) := \sum_{n=1}^\infty |\alpha_n| p(x_n)$$

defines a continuous semi-norm on E .

Further details can be found in [28, Section 3.3].

Lemma 7.3. *Let \mathcal{I} be an operator ideal. If $\{E_n\}_{n=1}^\infty$ is an \mathcal{S} -absolute decomposition of the locally convex space E , then E has the \mathcal{I} -approximation property if and only if each E_n has the \mathcal{I} -approximation property.*

Proof. An adaptation of the proof of [10, Proposition 1] works in this case. We give the details for the sake of completeness. Suppose that each E_n has the \mathcal{I} -approximation property. Let K denote a compact subset of E , let p denote an arbitrary continuous semi-norm on E satisfying condition (2) above with $\alpha_n = 1$ for all n , and let $\varepsilon > 0$ be arbitrary.

Define

$$\pi_n \left(\sum_{m=1}^{\infty} x_m \right) := \sum_{m=1}^n x_m$$

for all $\sum_{m=1}^{\infty} x_m \in E$ and let $\pi^n = id_E - \pi_n$. By [28, Lemma 3.33] there exists a positive integer n_0 such that

$$\sup\{p(\pi^{n_0}(x)) : x \in K\} < \varepsilon.$$

Consider $F_{n_0} := \pi_{n_0}(E) = \sum_{n=1}^{n_0} E_n$. By the locally convex version of Proposition 3.6 (see Remark 7.1) F_{n_0} has the \mathcal{I} -approximation property. Since $\pi_{n_0}(K)$ is a compact subset of F_{n_0} there exists an operator $T \in \mathcal{I}(F_{n_0}; F_{n_0})$ such that

$$p(\pi_{n_0}(x) - T(\pi_{n_0}(x))) < \varepsilon$$

for every $x \in K$. Using the natural inclusion $i: F_{n_0} \hookrightarrow E$ we see that $R := i \circ T \circ \pi_{n_0} \in \mathcal{I}(E; E)$. Hence

$$\begin{aligned} p(x - R(x)) &\leq p(x - \pi_{n_0}(x)) + p(\pi_{n_0}(x) - T(\pi_{n_0}(x))) \\ &= p(\pi^{n_0}(x)) + p(\pi_{n_0}(x) - T(\pi_{n_0}(x))) < 2\varepsilon \end{aligned}$$

for every $x \in K$. So $id_E \in \overline{\mathcal{I}(E; E)}^{\tau_c}$. It follows that E has \mathcal{I} -AP by the locally convex version of Proposition 3.1.

Conversely, since each E_n is a complemented subspace of E and E has the \mathcal{I} -approximation property, by the locally convex version of Proposition 3.5 it follows that each E_n has \mathcal{I} -approximation property as well. \square

By $\mathcal{P}_w(^n E)$ we mean the closed subspace of $\mathcal{P}(^n E)$ of all continuous n -homogeneous polynomials that are weakly continuous on bounded sets. Let U be an open subset of a Banach space E . A bounded subset A of U is U -bounded if there is a 0-neighborhood V such that $A + V \subseteq U$. By $\mathcal{H}_b(U; F)$ we denote the space of holomorphic functions $f: U \rightarrow F$, where F is a Banach space, of *bounded type*, that is, f is bounded on U -bounded sets. If $F = \mathbb{C}$ we simply write $\mathcal{H}_b(U)$. The symbol $\mathcal{H}_{wu}(U)$ stands for the space of all holomorphic functions $f: U \rightarrow \mathbb{C}$ that are weakly uniformly continuous on U -bounded sets. When endowed with the topology of uniform convergence on U -bounded sets, both $\mathcal{H}_b(U; F)$ and $\mathcal{H}_{wu}(U)$ are locally convex spaces.

Proposition 7.4. *Let \mathcal{I} be an operator ideal, U be a balanced open subset of the Banach space E and F be a Banach space.*

- (a) $\mathcal{H}_b(U; F)$ has \mathcal{I} -AP if and only if $\mathcal{P}(^n E; F)$ has \mathcal{I} -AP for every $n \in \mathbb{N}$.
- (b) $\mathcal{H}_{wu}(U)$ has \mathcal{I} -AP if and only if $\mathcal{P}_w(^n E)$ has \mathcal{I} -AP for every $n \in \mathbb{N}$.

Proof. Just combine Lemma 7.3 with the facts that $\{\mathcal{P}({}^n E; F)\}_{n=1}^\infty$ is an \mathcal{S} -absolute decomposition of $\mathcal{H}_b(U; F)$ (this follows from an adaptation of the proof of [28, Proposition 3.36]) and that $\{\mathcal{P}_w({}^n E)\}_{n=1}^\infty$ is an \mathcal{S} -absolute decomposition of $\mathcal{H}_{wu}(U; F)$ (see the proof of [10, Theorem 9]). \square

In the sequel some of our apparently disconnected results will be combined altogether. A Banach space E is said to be *polynomially reflexive* if $\mathcal{P}({}^n E)$ is reflexive for every $n \in \mathbb{N}$. For example, Tsirelson's original space T^* is polynomially reflexive [1].

Proposition 7.5. *Let \mathcal{I} be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$ and $\mathcal{I} \subseteq \mathcal{I}^{dual}$ or $\mathcal{I}^{dual} \subseteq \mathcal{I}$. The following are equivalent for a polynomially reflexive Banach space E and a balanced open subset U of E :*

- (a) E has \mathcal{I} -AP.
- (b) $\mathcal{P}({}^n E)$ has \mathcal{I} -AP for every $n \in \mathbb{N}$.
- (c) $\mathcal{P}({}^n E)$ has \mathcal{I} -AP for some $n \in \mathbb{N}$.
- (d) $\mathcal{H}_b(U)$ has \mathcal{I} -AP.

Proof. (a) \implies (b) Let $n \in \mathbb{N}$. By Corollary 5.9 we know that $\hat{\otimes}_\pi^{n,s} E$ has \mathcal{I} -AP. Since $\mathcal{P}({}^n E)$ is isomorphic to $(\hat{\otimes}_\pi^{n,s} E)'$ and these spaces are reflexive, by Corollary 4.3 we have that $\mathcal{P}({}^n E)$ has \mathcal{I} -AP.

(b) \implies (c) This implication is obvious.

(c) \implies (a) By [3, Proposition 5.3] it follows that E' is isomorphic to a complemented subspace of $\mathcal{P}({}^n E)$, so E' has \mathcal{I} -AP by Proposition 3.5. Then E has \mathcal{I} -AP by Corollary 4.3.

(d) \iff (b) This equivalence follows from Corollary 7.4(a). \square

To get another connection of the results from different sections we consider the predual of the space of holomorphic functions: given an open subset U of a Banach space, Mazet [42] proved the existence of a complete locally convex space $G(U)$ and of a canonical holomorphic function $\delta_U: U \rightarrow G(U)$ such that for every Banach space F and every holomorphic function from U to F there is a unique continuous linear operator T_f from $G(U)$ to F such that $f = T_f \circ \delta_U$.

Proposition 7.6. *Let U be a balanced open subset of the Banach space E and \mathcal{I} be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$. Then E has \mathcal{I} -AP if and only if $G(U)$ has \mathcal{I} -AP.*

Proof. For every $n \in \mathbb{N}$ let $Q({}^n E)$ be the space of all linear functionals φ on $\mathcal{P}({}^n E)$ such that the restriction of φ to each locally bounded subset is τ_c -continuous. By [9, Proposition 4] we have that $\{Q({}^n E)\}_{n=1}^\infty$ is an \mathcal{S} -absolute decomposition of $G(U)$, so, by Lemma 7.3, $G(U)$ has \mathcal{I} -AP if and only if $Q({}^n E)$ has \mathcal{I} -AP for every n . But $Q({}^n E)$ is isomorphic to $\hat{\otimes}_\pi^{n,s} E$ (see [9, p. 223]), so by Corollary 5.9 we have that $G(U)$ has \mathcal{I} -AP if and only if $Q({}^n E)$ has \mathcal{I} -AP for every n if and only if $\hat{\otimes}_\pi^{n,s} E$ has \mathcal{I} -AP for every n if and only if E has \mathcal{I} -AP. \square

The results from Section 6 have not been combined with results from other sections yet. For results of Section 6 to come into play we investigate the \mathcal{I} -approximation property in the predual of the space $\mathcal{H}^\infty(U; F)$ of bounded holomorphic functions from an open subset U of a Banach space E to a Banach space F . $\mathcal{H}^\infty(U; F)$ is a Banach space with the sup norm. Let U be an open subset of a Banach space E . Mujica [45] proved the existence of a Banach space $G^\infty(U)$ and of a canonical bounded holomorphic mapping $\delta_U \in \mathcal{H}^\infty(U; G^\infty(U))$ with the following universal property: to every $f \in \mathcal{H}^\infty(U; F)$ corresponds a unique linear operator $T_f \in \mathcal{L}(G^\infty(U); F)$ such that $f = T_f \circ \delta_U$. He also introduced a very useful locally convex topology on $\mathcal{H}^\infty(U; F)$:

Theorem 7.7. [45, Theorem 4.8] *Let E and F be Banach spaces, and let U be an open subset of E . Let τ_γ denote the locally convex topology on $\mathcal{H}^\infty(U; F)$ generated by the seminorms of the form*

$$p(f) = \sup_j \alpha_j \|f(x_j)\|,$$

where (x_j) varies over all sequences in U and (α_j) varies over all sequences of positive real numbers tending to zero. Then the mapping

$$f \in (\mathcal{H}^\infty(U; F), \tau_\gamma) \longrightarrow T_f \in (\mathcal{L}(G^\infty(U), \tau_c)$$

is a topological isomorphism.

We denote by $\mathcal{I} \circ \mathcal{H}^\infty(U; F)$ the collection of all $f \in \mathcal{H}^\infty(U; F)$ so that $f = u \circ g$, where G is a Banach space, $g \in \mathcal{H}^\infty(U; G)$ and $u \in \mathcal{I}(G; F)$. Next result extends [19, Theorem 5].

Theorem 7.8. *Let \mathcal{I} be an operator ideal such that $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$. The following conditions are equivalent for a Banach space E and an open subset U of E :*

- (a) E has \mathcal{I} -AP.
- (b) $\mathcal{H}^\infty(U; F) = \overline{\mathcal{I} \circ \mathcal{H}^\infty(U; F)}^{\tau_\gamma}$ for every Banach space F .
- (c) $G^\infty(U)$ has \mathcal{I} -AP.

Proof. (a) \implies (b) Let $f \in \mathcal{H}^\infty(U; F)$. Let p be a continuous semi-norm on $(\mathcal{H}^\infty(U; F), \tau_\gamma)$. By [45, Proposition 5.2] there are homogeneous polynomials $P_j \in \mathcal{P}^j(E; F)$, $j = 0, 1, \dots, n$, such that $p(P - f) < \frac{\varepsilon}{2}$ where $P = P_0 + P_1 + \dots + P_n$. Since E has \mathcal{I} -AP and $\mathcal{L}[\mathcal{I}] \subseteq \mathcal{I} \circ \mathcal{L}$, it follows from Proposition 6.2 that $\mathcal{P}^j(E; F) = \overline{\mathcal{I} \circ \mathcal{P}^j(E; F)}^{\tau_c}$ for every $j \in \mathbb{N}$. On the other hand, by [45, Proposition 4.9], $\tau_\gamma = \tau_c$ on $P \in \mathcal{P}^j(E; F)$ for every $j \in \mathbb{N}$. So there are homogeneous polynomials $Q_j \in \mathcal{I} \circ \mathcal{P}^j(E; F)$ such that

$$p(Q_j - P_j) < \frac{\varepsilon}{2(n+1)}$$

for every $j = 0, 1, \dots, n$. Putting $Q = Q_0 + Q_1 + \dots + Q_n$, mimicking an argument used in the proof of [2, Theorem 2.2] one can easily prove that Q is

of the form $Q = u \circ R$ where $u \in \mathcal{I}(E; G)$, G is a Banach space and R is a finite sum of homogeneous polynomials from G to F . Then the restriction of Q to U , still denoted by Q , is a bounded holomorphic function, so $Q \in \mathcal{I} \circ \mathcal{H}^\infty(U; F)$. Since

$$p(Q - P) = p\left(\sum_{j=0}^n Q_j - \sum_{j=0}^n P_j\right) \leq \sum_{j=0}^n p(Q_j - P_j) < \frac{\varepsilon}{2},$$

it follows that

$$p(Q - f) \leq p(Q - P) + p(P - f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which proves (b).

(b) \implies (c) By [45, Theorem 2.1], $\delta_U \in \mathcal{H}^\infty(U; G^\infty(U))$. Taking $F = G^\infty(U)$ in (b), we have that $\delta_U \in \overline{\mathcal{H}^\infty(U; G^\infty(U))}^{\tau_\gamma}$. Hence there is a net $(f_\alpha) \subseteq \mathcal{H}^\infty(U; G^\infty(U))$ such that $f_\alpha \xrightarrow{\tau_\gamma} \delta_U$. As to the corresponding net (T_{f_α}) of linear operators, by Theorem 7.7 we get that

$$T_{f_\alpha} \xrightarrow{\tau_c} T_{\delta_U} = id_{G^\infty(U)}.$$

But [2, Proposition 4.2] gives that $(T_{f_\alpha}) \subseteq \mathcal{I}(G^\infty(U); G^\infty(U))$. Therefore $id_{G^\infty(U)} \in \overline{\mathcal{I}(G^\infty(U); G^\infty(U))}^{\tau_c}$. By Proposition 3.4 we have that $G^\infty(U)$ has \mathcal{I} -AP.

(c) \implies (a) By [45, Proposition 2.3], E is topologically isomorphic to a complemented subspace of $G^\infty(U)$, which has \mathcal{I} -AP by assumption. It follows from Proposition 3.5 that E has \mathcal{I} -AP. \square

References

- [1] R. Alencar, R. Aron and S. Dineen, *A reflexive space of holomorphic functions in infinitely many variables*, Proc. Amer. Math. Soc. **90** (1984), 407–411.
- [2] R. Aron, G. Botelho, D. Pellegrino and P. Rueda *Holomorphic mappings associated to composition ideals of polynomials*, Rendiconti Lincei - Matematica e Applicazioni, to appear.
- [3] R. Aron and M. Schottenloher, *Compact holomorphic mappings on Banach spaces and the approximation property*, J. Funct. Anal. **21** (1976), 7–30.
- [4] S. Banach, *Théorie des opérations linéaires*, Warszawa, 1932.
- [5] F. Blasco, *Complementation in spaces of symmetric tensor products and polynomials*, Studia Math. **123** (1997), 165 – 173.
- [6] G. Botelho, *Ideals of polynomials generated by weakly compact operators*, Note Mat. **25** (2005/2006), 69–102.
- [7] G. Botelho, D. Pellegrino and P. Rueda, *On composition ideals of multilinear mappings and homogeneous polynomials*, Publ. Res. Inst. Math. Sci. **43** (2007), 1139–1155.

- [8] G. Botelho and P. Rueda, *The Schur property on projective and injective tensor products*, Proc. Amer. Math. Soc. **137** (2009), 219–225.
- [9] C. Boyd, *Distinguished preduals of spaces of holomorphic functions*, Rev. Mat. Univ. Complut. Madrid **6** (1993), 221–231.
- [10] C. Boyd, S. Dineen and P. Rueda, *Weakly uniformly continuous holomorphic functions and the approximation property*, Indag. Math. N.S. **12** (2001), 147–156.
- [11] H.-A. Braunss and H. Junek, *Factorization of injective ideals by interpolation*, J. Math. Anal. Appl. **297** (2004), 740–750.
- [12] B. Carl, A. Defant and M. S. Ramanujan, *On tensor stable operator ideals*, Michigan Math. J. **36** (1989), 63–75.
- [13] P. G. Casazza, *Approximation properties*, Handbook of the Geometry of Banach Spaces, Vol. I, 271–316, North-Holland, Amsterdam, 2001.
- [14] C. Choi and J. M. Kim, *Weak and quasi approximation properties in Banach spaces*, J. Math. Anal. Appl. **316** (2006), 722–735.
- [15] C. Choi and J. M. Kim, *On dual and three space problems for the compact approximation property*, J. Math. Anal. Appl. **323** (2006), 78–87.
- [16] C. Choi, J. M. Kim and K. Y. Lee, *Right and Left Weak Approximation Properties in Banach Spaces*, Canad. Math. Bull. **52** (2009), 28–38.
- [17] F. Cobos and I. Resina, *Representation theorems for some operator ideals*, J. London Math. Soc. (2) **30** (1989), 324–334.
- [18] E. Çaliskan, *Approximation of holomorphic mappings on infinite dimensional spaces*, Rev. Mat. Complut. **17** (2004), 411–434.
- [19] E. Çaliskan, *Bounded holomorphic mappings and the compact approximation property*, Port. Math. **61** (2004), 25–33.
- [20] E. Çaliskan, *Ideal of homogeneous polynomials and weakly compact approximation in Banach spaces*, Czechoslovak Math. J. **57(132)** (2007), 763–776.
- [21] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies 176, North-Holland, Amsterdam, 1993.
- [22] J. M. Delgado, E. Oja, C. Piñeiro and E. Serrano, *The p -approximation property in terms of density of finite rank operators*, J. Math. Anal. Appl. **354** (2009), 159–164.
- [23] J. M. Delgado, C. Piñeiro and E. Serrano, *Operators whose adjoints are quasi p -nuclear*, Studia Math. **197** (2010), 291–304.
- [24] J. M. Delgado, C. Piñeiro and E. Serrano, *Density of finite rank operators in the Banach space of p -compact operators*, J. Math. Anal. Appl., to appear.
- [25] J. Diestel, H. Jarchow and A. Pietsch, *Operator Ideals*, Handbook of the geometry of Banach spaces, Vol. I, 437–496, North-Holland, Amsterdam, 2001.
- [26] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators*, Cambridge University Press, 1995.
- [27] J. Diestel and J. J. Uhl, *Vector Measures*, American Mathematical Society, 1997.

- [28] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer Monographs in Math. Springer-Verlag, Berlin, 1999.
- [29] S. Dineen and J. Mujica, *The approximation property for spaces of holomorphic functions on infinite dimensional spaces I*, J. Approx. Theory **126** (2004), 141–156.
- [30] S. Dineen and J. Mujica, *The approximation property for spaces of holomorphic functions on infinite dimensional spaces II*, J. Funct. Anal. **259** (2010), 545–560.
- [31] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, Springer-Verlag, 2001.
- [32] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1995), 1–140.
- [33] J. R. Holub, *Tensor product mappings*, Math. Ann. **188** (1970), 1–12.
- [34] J. R. Holub, *Tensor product mappings II*, Proc. Amer. Math. Soc. **42** (1974), 437–441.
- [35] H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.
- [36] H. König, *On the tensor stability of s -number ideals*, Math. Ann. **269** (1984), 77–93.
- [37] G. Köthe, *Topological Vector Spaces II*, Springer-Verlag, 1979.
- [38] A. Lima, V. Lima and E. Oja, *Bounded approximation properties via integral and nuclear operators*, Proc. Amer. Math. Soc. **138** (2010), 287–297.
- [39] A. Lima and E. Oja, *The weak metric approximation property*, Math. Ann. **333** (2005) 471–484.
- [40] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II*, Springer, 1996.
- [41] A. Lissitsin, K. Mikkor and E. Oja, *Approximation properties defined by spaces of operators and approximability in operator topologies*, Illinois J. Math. **52** (2008), 563–582.
- [42] P. Mazet, *Analytic sets in locally convex spaces*, North-Holland Mathematics Studies 120, North-Holland, Amsterdam, 1986.
- [43] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, New York, 1998.
- [44] J. Mujica, *Complex Analysis in Banach Spaces*, North-Holland Math. Studies **120**. Amsterdam, 1986.
- [45] J. Mujica, *Linearization of bounded holomorphic mappings on Banach spaces*, Trans. Amer. Math. Soc. **324** (1991), 867–887.
- [46] J. Mujica, *Spaces of holomorphic functions and the approximation property*, IMI Graduate Lecture Notes **1**, Universidad Complutense de Madrid, 2009.
- [47] E. Oja, *Lifting bounded approximation properties from Banach spaces to their dual spaces*, J. Math. Anal. Appl. **323** (2006), 666–679.

- [48] E. Oja, *The strong approximation property*, J. Math. Anal. Appl. **338** (2008), 407–415.
- [49] A. Pełczyński, *On weakly compact polynomial operators on B -spaces with Dunford-Pettis Property*, Bull. Polish Acad. Sci. Math. **11** (1963), 371–377.
- [50] A. Pietsch, *Operator Ideals*, North-Holland, 1980.
- [51] A. Pietsch, *Tensor products of sequences, functions and operators*, Arch. Math. **38** (1982), 335–344.
- [52] G. Racher, *On the tensor product of weakly compact operators*, Math. Ann. **294** (1992), no. 2, 267–275.
- [53] D. P. Sinha, A. K. Karn, *Compact operators whose adjoints factor through subspaces of ℓ_p* , Studia Math. **150** (2002), 17–33.
- [54] A. Szankowski, *Subspaces without approximation property*, Israel J. Math. **30** (1978), 123–129.
- [55] G. Willis, *The compact approximation property does not imply the approximation property*, Studia Math. **103** (1992), 99–108.

Faculdade de Matemática
 Universidade Federal de Uberlândia
 38.400-902 - Uberlândia - Brazil
 e-mails: soniles@famat.ufu.br, botelho@ufu.br